THE SELF-SIMILAR PROBLEM OF IMPACT LOADING IN A NONLINEARLY ELASTIC MATERIAL

PMM Vol. 31, No. 5, 1967, pp. 877-882
E. M. CHERNYKH
(Voronezh)
(Received March 24, 1967)

Dynamic problems of stress in nonlinearly elastic (with infinitesimally small deformations) and elastic-plastic media have been investigated in [1 to 5].

Below, Eulerian coordinates are used to study the plane problem of a suddenly loaded half-space occupied by a geometrically nonlinear Almansi material (with finite deformations) [6]. Conditions permitting the construction of continuous and discontinuous self-similar solutions are established. A solution is obtained to the one-dimensional problem. An approximate solution is constructed to the problem of a half-space subjected to a sudden load which is inclined to its boundary. It is shown that an Almansi medium exhibits a Poynting effect [7 and 8].

Consider the half-space $x_1 \geq 0$ occupied by an Almansi medium [6], with the defining Eqs. given by

$$\sigma_{ij} = \lambda\delta_{ij}e_{kk} + 2\mu e_{ij}, \qquad e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right) \tag{1}$$

Assume that the medium is in an unstressed state up to the time $t = 0$. Suppose that at the instant $t = 0$ all points on the boundary plane $x_1 = 0$ are suddenly subjected to a finite constant motion with velocities v_{10}, v_{20}. Thus, following conditions will be satisfied on the moving boundary:

$$v_1(x_{10}) = v_{10}, \qquad v_2(x_{10}) = v_{20}, \qquad v_3 \equiv 0 \qquad (x_{10} = v_{10}t) \tag{2}$$

It is required to determine the state of stress resulting from the sudden load in (2).

The problem formulated permits certain assumptions with regard to the displacements

$$u_1 = u_1(x_1, t), \qquad u_2 = u_2(x_1, t), \qquad u_3 \equiv 0 \tag{3}$$

The equations of motion may be written, in Eulerian coordinates, as

$$\rho\left(\frac{\partial v_i}{\partial t} + v_k\frac{\partial v_i}{\partial x_k}\right) = \frac{\partial \sigma_{ij}}{\partial x_j} \tag{4}$$

$$v_i = \frac{\partial u_i}{\partial t} + v_k\frac{\partial u_i}{\partial x_k}, \qquad \frac{\partial u_i}{\partial t} = v_k\left(\delta_{ik} - \frac{\partial u_i}{\partial x_k}\right) \tag{5}$$

Differentiating (5) with respect to x_j, we obtain

$$\frac{\partial}{\partial t}\left(\frac{\partial u_i}{\partial x_j}\right) = \frac{\partial v_k}{\partial x_j}\left(\delta_{ik} - \frac{\partial u_i}{\partial x_k}\right) - v_k\frac{\partial^2 u_i}{\partial x_k \partial x_j} \tag{6}$$

The continuity equation, in Lagrangian form, is given by

$$\rho = \rho_0(1 - 2J_1 + 4J_2 - 8J_3)^{1/2} \tag{7}$$

Here, ρ_0 is the density in the unstressed state, and J_k are the invariants of the tensor e_{ij}.

Taking into account (1),(3) and (7), Eqs. (4) and (6) may be written as

$$\rho_0(1-\omega_1)\left(\frac{\partial v_1}{\partial t}+v_1\frac{\partial v_1}{\partial x_1}\right)=(\lambda+2\mu)\left[(1-\omega_1)\frac{\partial\omega_1}{\partial x_1}-\omega_2\frac{\partial\omega_2}{\partial x_1}\right] \tag{8}$$

$$\rho_0(1-\omega_1)\left(\frac{\partial v_2}{\partial t}+v_1\frac{\partial v_2}{\partial x_1}\right)=\mu\frac{\partial\omega_2}{\partial x_1}\qquad\left(\frac{\partial u_1}{\partial x_1}=\omega_1,\ \frac{\partial u_2}{\partial x_1}=\omega_2\right)$$

$$\frac{\partial\omega_1}{\partial t}=\frac{\partial v_1}{\partial x_1}(1-\omega_1)-v_1\frac{\partial\omega_1}{\partial x_1},\qquad\frac{\partial\omega_2}{\partial t}=-\omega_2\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_1}-v_1\frac{\partial\omega_2}{\partial x_1}$$

We introduce the nondimensional variables τ, x, V_1, V_2, defined by the relations

$$t=\tau T,\ x_1=xc_{10}T,\qquad v_1=V_1c_{10},\qquad v_2=V_2c_{10},\qquad (c_{10}^2=\lambda+2\mu/\rho_0) \tag{9}$$

Here T is a characteristic time.

Eqs. (8) may now be written in nondimensional form

$$(1-\omega_1)\left(\frac{\partial V_1}{\partial\tau}+V_1\frac{\partial V_1}{\partial x}\right)=(1-\omega_1)\frac{\partial\omega_1}{\partial x}-\omega_2\frac{\partial\omega_2}{\partial x}$$

$$(1-\omega_1)\left(\frac{\partial V_2}{\partial\tau}+V_1\frac{\partial V_2}{\partial x}\right)=k^2\frac{\partial\omega_2}{\partial x},\qquad k^2=\frac{\mu}{\lambda+2\mu} \tag{10}$$

$$\frac{\partial\omega_1}{\partial\tau}=\frac{\partial V_1}{\partial x}(1-\omega_1)-V_1\frac{\partial\omega_1}{\partial x},\qquad\frac{\partial\omega_2}{\partial\tau}=-\omega_2\frac{\partial V_1}{\partial x}+\frac{\partial V_2}{\partial x}-V_1\frac{\partial\omega_2}{\partial x}$$

The boundary conditions (2) become

$$V_1(x_0(\tau))=V_{10},\qquad V_2(x_0(\tau))=V_{20} \tag{11}$$

Since the coefficients in (10) do not contain x and τ explicitly, and V_{10}, V_{20} in (11) are constant), then (10) admits self-similar solutions when all the unknown quantities are functions of the "space-time coordinate" ratio ξ

$$\xi=\frac{x}{\tau},\qquad\frac{\partial}{\partial x}=\frac{1}{\tau}\frac{d}{d\xi},\qquad\frac{\partial}{\partial\tau}=-\frac{\xi}{\tau}\frac{d}{d\xi} \tag{12}$$

Application of (12) transforms (10) into the ordinary differential Eqs.

$$(1-\omega_1)(V_1-\xi)\frac{dV_1}{d\xi}-(1-\omega_1)\frac{d\omega_1}{d\xi}+\omega_2\frac{d\omega_2}{d\xi}=0,\ (1-\omega_1)(V_1-\xi)\frac{dV_2}{d\xi}-k^2\frac{d\omega_2}{d\xi}=0 \tag{13}$$

$$(1-\omega_1)\frac{dV_1}{d\xi}-(V_1-\xi)\frac{d\omega_1}{d\xi}=0,\qquad\omega_2\frac{dV_1}{d\xi}-\frac{dV_2}{d\xi}+(V_1-\xi)\frac{d\omega_2}{d\xi}=0$$

Eqs. (13) have a trivial solution when V_1, V_2, ω_1 and ω_2 are all constant. Here, the pertinent solutions have discontinuities along the ray ξ_* and are constant on either one or both sides of the ray. The nontrivial solutions will occur when the determinant of the coefficients in (13) vanishes.

Expanding and setting the determinant equal to zero, we obtain

$$\xi=V_1\pm C_{1,2} \tag{14}$$

$$C_{1,2}=\left\{\frac{(1-\omega_1)^2+\omega_2^2+k^2\pm\sqrt{[(1-\omega_1)^2+\omega_2^2-k^2]^2+4k^2\omega_2^2}}{2(1-\omega_1)}\right\}^{1/2} \tag{15}$$

Thus, the nontrivial solutions will occur on the fan of rays radiating from the origin of the $x\tau$ plane and inclined to the τ-axis at an angle whose tangent is given in (14). For the waves travelling in the $x>0$ region, the upper sign must be used in (14).

Substituting (14) into (13), we obtain the following systems of equations for the determination of the desired parameters:

on the first fan

$$(C_1^2 - 1 + \omega_1)\, d\omega_1 + \omega_2 d\omega_2 = 0$$

$$(1 - \omega_1)\, dV_1 + C_1 d\omega_1 = 0$$

$$dV_2 - \omega_2 dV_1 + C_1\, d\omega_2 = 0$$

on the second fan

$$(C_2^2 - 1 + \omega_1)\, d\omega_1 + \omega_2 d\omega_2 = 0$$

$$(1 - \omega_1)\, dV_1 + C_2\, d\omega_1 = 0$$

$$dV_2 - \omega_2 dV_1 + C_2\, d\omega_2 = 0$$

Consider the problem of constructing continuous solutions. Continuous solutions (if such are admissible under boundary conditions (11)), are constructed from the constant solution (zero) in the G_∞ region of the $x\tau$ plane (Fig. 1). A continuous transition to the variable solution in the G_1 region occurs on the ray OE. Here, solution is obtained by integrating (16) and utilizing conditions in the G_∞ region

$$V_{1\infty} = V_{2\infty} = \omega_{1\infty} = \omega_{2\infty} = 0 \qquad (17)$$

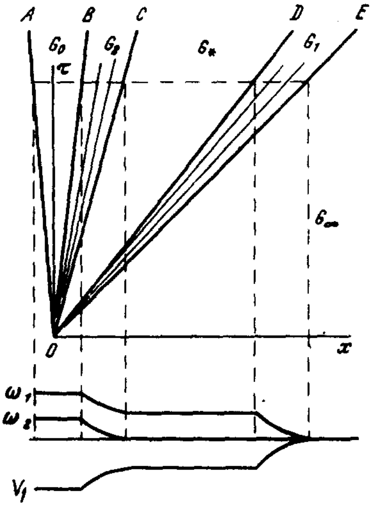

Fig. 1

This solution determines the quantities V_{1*}, V_{2*}, ω_{1*} and ω_{2*} on the ray OD. Note that from (15) $C_1 > C_2$ for all values of ω_1, ω_2. Thus, the ray OC, bounding the second fan on the right side, is always to the left on the ray OD, bounding the first fan on the left side. In the G region between these rays, the solution is constant: V_{1*}, V_{2*}, ω_{1*}, ω_{2*}. The solution on the second fan varies continuously in accordance with the Eqs. given in (16) for the second fan, and there is a continuous transition from this solution to the constant one of V_{10}, V_{20} ω_{10}, ω_{20} in the G_0 region. Note that the six conditions (11) and (17) completely define the six constants ω_{10}, ω_{20}, V_{1*}, V_{2*}, ω_{1*} and ω_{2*}, and hence the solution in G_1 and G_2.

It should be noted that the continuous solutions thus obtained must satisfy the conditions

$$\xi_{2*} = V_{1*} + C_{2*} \geqslant V_{10} + C_{20} = \xi_{20}, \qquad \xi_{1*} = V_{1*} + C_{1*} \leqslant V_{1\infty} + C_{1\infty} = \xi_{1\infty} \qquad (18)$$

Otherwise, no continuous solution is possible.

The construction of a solution in the presence of discontinuities follows a different procedure. It was shown in [10] that compressive shock waves can exist in an Almansi medium only in the direction of wave propagation. For the problem at hand, this means that ω_2 and V_2 will possess no discontinuity on the shock wave, while the discontinuity $[\omega_1] = \omega_{*1} < 0$.

The following relations hold for the dimensionless propagation velocities of shock waves on the different sides of the discontinuity:

$$\theta_* = [(1 - {}^1\!/_2\omega_{1*})(1 - \omega_{1*})^{-1}]^{1/2}, \qquad V_{1*} = \theta_0 - \theta_* \qquad (19)$$

$$\theta_0 = [(1 - {}^1\!/_2\omega_{1*})(1 - \omega_{1*})]^{1/2},$$

The trajectory of the shock wave in the $x\tau$ plane is given by the ray OD which is inclined to the τ-axis at an angle whose tangent is equal to θ_0 (Fig. 2). In the G_∞ region, the solution is equal to zero (17). On the shock wave OD, the quantities V_1 and ω_1 have discontinuities. Because of the presence of the shock wave, the first fan of characteristics vanishes. Everywhere in the region G_*, bounded by the shock wave and the characteristic OC constituting the right side boundary of the second fan, the following solution is constant:

$$V_1 = V_{1*} \neq 0, \qquad \omega_1 = \omega_{1*} \neq 0, \qquad \omega_{2*} = V_{2*} = 0$$

The values of the constants V_{1*} and ω_{1*} are not known beforehand, but are determined from (16) and the boundary conditions (11) in the region G_0. The relation between V_{1*} and ω_{1*} is obtained from (19). The solution in the region to the left of OC is constructed in a manner similar to that described above.

A straightforward analysis of (16) and (19) shows that, in both cases, the resultant solutions yield the functional dependence of $\omega_{1*}, \omega_{2*}, V_{1*}, V_{*2}, \omega_{10}$ and ω_{20} on V_{10} and V_{20} whereupon the solutions in the regions G_1 and G_2 may be constructed.

However, these functional relations are not generally obtainable for arbitrary V_{10} and V_{20}, since the system of differential equations (16) is not integrable by quadrature. The study below gives an exact solution for the one-dimensional case for which $V_{20} = 0$ and an approximate solution for the general case of an oblique shock, for small V_{10} and V_{20}.

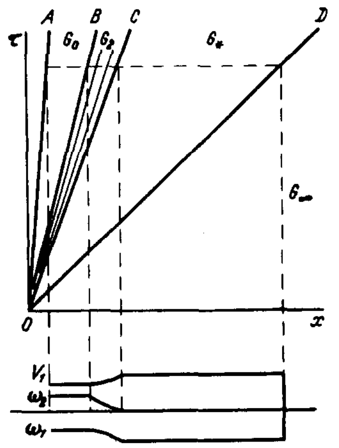

Fig. 2

One-dimensional problem. In this case, $\omega_2 = V_2 \equiv 0$. From (15), $C_1 = (1 - \omega_1)^{1/2}$, $C_2 = 0$. The only possible solution to (16) is the constant one, i.e. there is no second fan G_2 (Fig. 1), and the parameters in the regions G_0 and G_* are identical. The first and third Eqs. in (16), for the first fan are identically satisfied while the second equation is easily integrated. The resultant solution in the G_1 region is

$$V_1 - V_{10} = 2\,[(1 - \omega_1)^{1/2} - (1 - \omega_{10})^{1/2}],$$
$$\xi = V_1 + (1 - \omega_1)^{1/2} \qquad (20)$$

Applying conditions (17) on the ray OE, we obtain from (20)

$$\omega_{10} = -V_{10}\,(1 + {}^1/_4 V_{10}) \qquad (21)$$

From the combination of (20) and (21) we obtain the following explicit relations between V_1, ω_1 and ξ, which are valid under conditions obtainable from (18):

$$(V_1 = {}^2/_3(\xi - 1),\ \omega_1 = 1 - {}^1/_9(\xi + 2)^2,\ 1 + {}^3/_2 V_{10} = \xi_* \leqslant \xi \leqslant 1,\ V_{10} \leqslant 0$$

where ξ_* and ξ_0 are boundary values of ξ on the rays OD and OE, respectively.

If conditions (18) are not satisfied, i.e. $V_{10} > 0$, then the solution is piecewise constant but discontinuous. The functional dependence of ω_{10} and θ on $V_{10} = V_{1*}$ is given by (20).

Oblique shock for small V_{10} and V_{20}. In this case, $\omega_1(x, \tau) \ll 1$, $\omega_2(x, \tau) \ll 1$. The coefficients in (16) may be expanded into series in ω_1 and ω_2, and self-similar solutions may be constructed approximately in terms of series. Hereinafter, we confine ourselves to a first approximation, neglecting all quantities of a higher order of smallness compared to unity.

The justification for such an approximation is that, in the one-dimensional case, the results of such an approximation were sufficiently close to the exact solution (for very strong shock waves with stress discontinuities of the order of 10^6 kg/cm^2 ($\omega_{10} \approx 0.1$) the discrepancies in calculations did not exceed 1%).

Investigation of the continuous solution. For the velocity of sound (15), we have the approximate Expressions

$$C_1^2 = 1 - \omega_1,\ C_1 = 1 - {}^1/_2\,\omega_1; \qquad C_2^2 = k^2(1+\omega_1),\ C_2 = k(1 + {}^1/_2\,\omega_1) \qquad (22)$$

Integration of (16), making use of the conditions in the regions G_∞ and G_* (Fig. 1), yields the solution for the region G_1

$$\omega_2 = \omega_{2_*} = 0, \qquad V_2 = V_{2_*} = 0 \qquad (23)$$

$$V_1 - V_{1_*} = \omega_{1_*} - \omega_1 = {}^1/_4\omega_{1*} - {}^1/_4\omega_1^2, \qquad \xi_1 = V_1 + 1 - {}^1/_2\omega_1$$

Integration of (16) for the second fan, making use of the conditions in region G_*, yields the solution for the region G_2

$$V_1 - V_{1_*} = k\,[\omega_{1*} - \omega_1 + {}^3/_4(\omega_{1*}^2 - \omega_1^2)] \qquad (24)$$

$$\omega_2^2 = 2(1-k^2)(\omega_1 - \omega_{1_*}) - (k^2+1)(\omega_1^2 - \omega_{1_*}^2), \qquad \xi_2 = V_1 + k(1 + {}^1/_2\omega_1)$$

$$V_2^2 = 2k^2(1-k^2)(\omega_1 - \omega_{1*})\left[1 + \frac{7-13k^2}{6(1-k^2)}\omega_1 - \frac{7-k^2}{6(1-k^2)}\,\omega_{1*}\right]$$

To determine ω_{1_*}, V_{1_*}, ω_{10} and ω_{20}, it is necessary to set $V_1 = \omega_1 = 0$ in (23) (region G_∞), and to set $V_1 = V_{10}$, $V_2 = V_{20}$, $\omega_1 = \omega_{10}$ and $\omega_2 = \omega_{20}$ in (24) (region G_0). The approximate resultant solution is given by

$$\omega_{10} = -V_{10} + a_{10}V_{10}^2 + a_{20}V_{20}^2, \qquad \omega_{1*} = -V_{10} + a_{1_*}V_{10}^2 + a_{2_*}V_{20}^2 \qquad (25)$$

$$V_{1_*} = V_{10} + b_1V_{10}^2 + b_2V_{20}^2$$

$$a_{10} = \frac{9k^6 - 30k^5 + 20k^4 + 20k^3 - 8k^2 - 12k - 17}{12(1-k)(1-k^2)}$$

$$a_{20} = [2k^2(1+k)]^{-1}, \qquad a_{1_*} = -{}^1/_4(1-k)(1-3k) - k({}^3/_4 + a_{10})/(1-k)$$

$$a_{2_*} = -[2k(1-k^2)]^{-1}, \qquad b_1 = -({}^1/_4 + a_{1_*}), \qquad b_2 = -a_{2*}$$

Utilizing (18), we obtain the conditions which must be satisfied by V_{10} and V_{20} in order that a solution be possible. Since, in virtue of the second Eq. in (24), $\omega_1 - \omega_{1_*} \geq 0$ for any ω_1, it is easily seen that the first inequality in (18) holds everywhere

$$\xi_{2*} - \xi_{20} = d_1V_{10}^2 + d_2V_{20}^2 \geqslant 0 \qquad (26)$$

$$d_1 = b_1 - a_{1_*} + {}^1/_2k\,(a_{1_*} - a_{10}), \qquad d_2 = b_2 - a_{2_*} + {}^1/_2k\,(a_{2*} - a_{20})$$

The second relation in (18) may be brought into the form

$$\xi_{1*} - \xi_{1\infty} = {}^3/_2\,[V_{10} + {}^1/_3({}^1/_2 + a_{1_*})V_{10}^2 + {}^1/_2k^{-1}(1-k^2)^{-1}V_{20}^2] \leqslant 0 \qquad (27)$$

From (26) and (27), it follows that the differences $\xi_1 - \xi_{1\infty}$ and $\xi_2 - \xi_{20}$ are small. Expanding (23) and (24) in terms of these differences and retaining the low-order terms, we obtain expressions for V_1, V_2, ω_1 and ω_2 explicitly depending on the direction of the rays in G_1 and G_2.

In G_1, the solution is given by

$$\omega_1 = -{}^2/_3(\xi - 1) - {}^2/_{27}(\xi - 1)^2 \qquad (28)$$

$$V_1 = -\omega_1(1 + {}^1/_4\,\omega_1), \qquad \omega_2 = V_2 = 0, \qquad \xi_{1*} \leqslant \xi \leqslant 1$$

with ξ_{1*} as given by (27).

In G_2, the solution is given by

$$\omega_1 = \omega_{10} + (k)^{-1}\,({}^3/_2\,\omega_{10} - 1)\,(\xi - \xi_{20}) - {}^3/_4\,k^{-2}(\xi - \xi_{20})^2, \quad \xi_{20} \leqslant \xi \leqslant \xi_{2*} \qquad (29)$$

The functions ω_2, V_1 and V_2 are determined from (24) and (29), making use of (25), with ξ_{2*} as given by (26).

Fig. 1 represents the calculated results as functions of x at a fixed time T_1. At any other time T_2, the curves will be similarly stretched along the x-axis at a ratio of T_2/T_1.

If the initial values V_{10} and V_{20} do not satisfy (27), the preceding solution loses meaning. It is of interest to note that (27) is not satisfied for a pure shear ($V_{10} = 0$) along the half-space boundary, i.e. a sudden shear at the boundary causes a compressive shock wave in a direction normal to the boundary. This represents a dynamic manifestation of the Poynting effect [7 and 8].

An approximate, discontinuous solution is constructed in the following manner. From (19), we have

$$V_{1*} = -\omega_{1*}(1 + {}^1/_4\,\omega_{1*}), \qquad \theta_0 = 1 - {}^3/_4\,\omega_{1*} - {}^1/_8\,\omega^2_{1*} \qquad (30)$$

Eqs. (24) and (26) are still satisfied. In the G_∞ region, the solution equals to zero. On the ray OD (Fig. 2), the quantities V_1 and ω_1 have jumps from zero to V_{1*} and ω_{1*}, respectively, obtained from (25), while $V_{2*} = \omega_{2*} = 0$. In the G_2 region, (29) holds. The resulting solutions are shown in Fig. 2.

Note the following. As (26) shows, the angular range of the characteristics of the second fan is a second-order quantity in comparison with V_{10} and V_{20}. On this very "narrow" fan, the change in the state of stress is very intense, albeit continuous. However, the region in which this change occurs increases in direct proportion to the elapsed time. Thus, the initial discontinuity in the shear velocity V_{20} gradually diffuses through the medium. This is the structure of a shearing "shock" wave in an Almansi medium.

The known [11 and 12] results for a linearly elastic, Hookean material are obtained here as the limiting case for $V_{10} \to 0$, $V_{20} \to 0$.

The unloading problem is not examined here.

BIBLIOGRAPHY

1. Antsiferov, V. S. and Rakhmatulin, Kh. A., The propagation of compres-sure-shear perturbations in a nonlinearly elastic medium. PMM Vol. 28, N°3, 1964.
2. Bykovtsev, G. I., On the propagation of disturbances in media with nonlinear stress-strain relations. PMTF, No. 4, 1961.
3. Pei Chi, Chou and Koenig, H. A., A unified approach to cylindrical and spherical elastic waves by method of characteristics. J. Appl. Mech., Vol. 33, No. 1, 1966.
4. Bleich, H. H. and Nelson, I., Plane waves in an elastic plastic half-space due to combined surface pressure and shear. J. Appl. Mech., Vol. 33, No. 1, 1966.
5. Clifton, R. J. and Bodner, S. R., An analysis of longitudinal elastic-plastic pulse propagation. J. Appl. Mech., Vol. 33, No. 2.
6. Prager, W., Einführung in die Kontinuumsmechanik. Basel-Stuttgart, 1961.
7. Reiner, M., Rheology, Berlin-Göttingen-Heidelberg, 1958.
8. Freudenthal, A. M. and Geiringer, H., The Mathematical Theories of the Inelastic Continuum. Berlin-Göttingen-Heidelberg, 1958.
9. Sedov, L. I., Introduction to the Mechanics of Continuous Media. M., Fizmatgiz, 1962.
10. Chernykh, E. M., On wave propagation in an elastic medium with finite deformations. Inzh. zh. MTT, No. 4, 1967.
11. Kolsky, H., Stress Waves in Solids. Oxford, 1953.
12. Thomas, T., Plastic Flow and Fracture in Solids. New York, London, 1961.

Translated by H. H.